Almost partitioning 2-edge-colourings of 3-uniform hypergraphs with two monochromatic tight cycles

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Abstract
We show that any 2-colouring of the 3-uniform complete hypergraph $K^{(3)}_n$ on $n$ vertices contains two disjoint monochromatic tight cycles of distinct colours covering all but $o(n)$ vertices of $K^{(3)}_n$. The same result holds if we replace tight cycles with loose cycles.

Keywords: Monochromatic cycle partitioning, tight cycles.

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1 Introduction

Given a complete $r$-edge-colouring of graph or hypergraph $\mathcal{K}$, the problem of partitioning the vertices of $\mathcal{K}$ into the smallest number of monochromatic cycles has received much attention. Central to this area has been an old conjecture of Lehel [2] stating that two monochromatic disjoint cycles in different colours are sufficient to partition the vertex set of the complete graph $\mathcal{K}_n$ on $n$ vertices, for all $n$. This was confirmed for large $n$ in [10] and [1], and more recently, for all $n$, by Bessy and Thomassé [3].

For $r \geq 3$, there exist $r$-edge-colourings of $\mathcal{K}_n$ which do not allow for a partition of the vertex set into $r$ monochromatic cycles [11]. On the other hand, the currently best bound (see [6]) shows that $100r \log r$ monochromatic cycles are sufficient to partition the vertex set of $\mathcal{K}_n$.

The problem transforms in the obvious way to hypergraphs, considering $r$-edge-colourings of the $k$-uniform complete hypergraph $\mathcal{K}_n^{(k)}$ on $n$ vertices and partitions into one of the many notions of cycles in hypergraphs. Here we deal with loose and tight cycles. Loose cycles are uniform hypergraphs with a cyclic ordering of its edges such that consecutive edges intersect in exactly one vertex and nonconsecutive edges have empty intersection. On the other hand, tight cycles are $k$-uniform hypergraphs with a cyclic ordering of its vertices such that the edges are all the sets of $k$ consecutive vertices. For loose cycles, the best bound due to Sárközy in [12] shows that every $r$-edge-colouring of $\mathcal{K}_n^{(k)}$ admits a partition of its vertices into at most $50rk \log(rk)$ monochromatic loose cycles. Concerning tight cycles, to our best knowledge, nothing is known. We refer the reader to [5] for related results.

Our main result establishes an approximate version of the problem for the case of 3-uniform hypergraphs and two colours.

**Theorem 1.1** For every $\eta > 0$ there exists $n_0$ such that if $n \geq n_0$ then every 2-coloring of the edges of the complete 3-uniform hypergraph $\mathcal{K}_n^{(3)}$ admits two vertex-disjoint monochromatic tight cycles, of distinct colours, which cover all but at most $\eta n$ vertices.

We note that a 3-uniform tight cycle on $n$ vertices contains a loose cycle if $n$ is even. The proof of Theorem 1.1 guarantees that the two tight cy-

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cycles obtained each have an even number of vertices. Hence, an analogue of Theorem 1.1 holds for loose cycles.

We believe that the error term \( \eta n \) in the theorem can be improved and that every 2-colouring of the edges of \( K_n^{(3)} \) admits two disjoint monochromatic tight cycles which cover all but at most a constant number \( c \) of vertices (for some \( c \) independent of \( n \)). Furthermore, we believe that the previous statement holds for all \( k \) and not just \( k = 3 \). In a forthcoming article we confirm this for loose cycles, where the constant \( c \) depends only on \( k \).

2 Outline of the proof

Due to lack of space we only give a sketch of the argument, referring to [4] for full details for the proof of Theorem 1.1.

The argument is inspired by the work of Haxell et al. [8] and relies on an application of Łuczak’s method [9]. This reduces the problem at hand to that of finding, in any 2-colouring of the edges of an almost complete 3-uniform hypergraph, two disjoint monochromatic connected matchings which cover almost all vertices.

Here, as usual, a matching \( \mathcal{M} \) in a hypergraph \( \mathcal{H} \) is a set of pairwise disjoint edges and \( \mathcal{M} \subset \mathcal{H} \) is called connected if between every pair \( e, f \in \mathcal{M} \) there is a pseudo-path in \( \mathcal{H} \) connecting \( e \) and \( f \), that is, there is a sequence \((e_1, \ldots, e_p)\) of not necessarily distinct edges of \( \mathcal{H} \) such that \( e = e_1, f = e_p \) and \( |e_i \cap e_{i+1}| = 2 \) for each \( i \in [p-1] \). Now, we call a matching \( \mathcal{M} \) in a 2-coloured hypergraph a monochromatic connected matching if \( \mathcal{M} \) is a subhypergraph of a monochromatic component induced by the considered relation of connectedness.

Our main result is the following, which might be of independent interest.

**Theorem 2.1** Let \( \mathcal{H} \) be a 3-uniform hypergraph on \( t \) vertices and \((1 - \gamma){t \choose 3}\) edges. Then any two-colouring of the edges of \( \mathcal{H} \) admits two disjoint monochromatic connected matchings covering all but \( o(t) \) vertices of \( \mathcal{H} \).

We first give an outline of the proof of Theorem 1.1 assuming that Theorem 2.1 holds, before dealing with Theorem 2.1 itself.

2.1 Proof of Theorem 1.1

For given \( \eta > 0 \) we apply the Strong Regularity Lemma (see [8] for details) for 3-uniform hypergraphs to \( K_n^{(3)} \) with suitable parameters to obtain a regular partition and the reduced hypergraph \( \mathcal{K} \) on \( t \) vertices and \((1 - \gamma){t \choose 3}\)
edges, where $\gamma$ depends on $\eta$. Consider the 2-edge-colouring of $K$ given by the majority colouring over the triples of the regular partition.

Next, apply Theorem 2.1 to $K$ to obtain the monochromatic connected matchings $M_{\text{red}}$ and $M_{\text{blue}}$ covering all but $o(t)$ vertices of $K$.

By using $M_{\text{red}}$ and $M_{\text{blue}}$ as a frame and applying a suitable embedding strategy (see [4]) we find find two monochromatic disjoint tight cycles of even length covering at least $(1 - \eta)n$ vertices of $K^{(3)}$, as desired.

2.2 Proof of Theorem 2.1

We will need the following result concerning the existence of perfect matchings in 3-uniform hypergraphs with high minimum vertex degree.

Theorem 2.2 ([7]) For all $\eta > 0$ there is a $n_0 = n_0(\eta)$ such that for all $n > n_0$, $n \in 3\mathbb{Z}$, the following holds. Suppose $H$ is 3-uniform hypergraph on $n$ vertices such that every vertex is contained in at least $(\frac{5}{9} + \eta)\binom{n}{2}$ edges. Then $H$ contains a perfect matching.

Denote by $\partial H$ the shadow of $H$, that is, the set of all pairs $xy$ for which there exists $z$ such that $xyz \in H$. We call a pair of vertices $xy$ active if there is an edge of $H$ containing $x$ and $y$. For convenience, we say that a set of vertices $U \subseteq V(H)$ is negligible in $H$ if $|U| \leq 240\gamma^{1/6}|V(H)|$.

Lemma 2.3 ([8]) Let $\gamma > 0$ and let $H$ be a 3-uniform hypergraph on $t$ vertices and at least $(1 - \gamma)\binom{t}{3}$ edges. Then $H$ contains a subhypergraph $K$ such that the following holds. Every vertex $x$ of $K$ is in an active pair of $K$, for all active pairs $xy$ there are at least $(1 - 10\gamma^{1/6})t$ edges in $K$ containing both $x$ and $y$, and $V(H) \setminus V(K)$ is negligible in $H$.

For our proof of Theorem 2.1, suppose we are given a 2-coloured 3-uniform hypergraph $H = H_{\text{red}} \cup H_{\text{blue}}$ on $t_H$ vertices and $(1 - \delta)\binom{t_H}{3}$ edges. Apply Lemma 2.3 to $H$, with parameter $\gamma$ depending on $\delta$, to obtain $K$ with the properties stated in the lemma. We want to find two monochromatic connected matchings covering all but a negligible set of vertices in $K$. Let $K = K_{\text{red}} \cup K_{\text{blue}}$ be the colouring of $K$ inherited from $H$.

Proposition 2.4 ([8]) The hypergraph $K$ admits a partition $\{X, V_{\text{red}}, V_{\text{blue}}\}$ such that the following holds. The set $X$ is negligible in $K$ and there is a red component $R$ (a blue component $B$) such that, for every $x \in V_{\text{red}}$ ($x \in V_{\text{blue}}$), there are at least $(1 - \gamma)t$ vertices $y \in V(K)$ with $xy \in \partial R$ ($xy \in \partial B$).

We start by choosing two disjoint monochromatic connected matchings,
\[ \mathcal{M}_{\text{red}} \subseteq \mathcal{R} \text{ and } \mathcal{M}_{\text{blue}} \subseteq \mathcal{B}, \text{ where } \mathcal{R} \text{ and } \mathcal{B} \text{ are components from Proposition 2.4, which together cover as many vertices as possible. Let } V'_{\text{red}} = V_{\text{red}} \setminus (V(\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}) \text{ and } V'_{\text{blue}} = V_{\text{blue}} \setminus (V(\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}})). \text{ Notice that if both } V'_{\text{red}} \text{ and } V'_{\text{blue}} \text{ are negligible in } \mathcal{K} \text{ we are done. Also, observe that there is no edge } xy \text{ with } x \in V'_{\text{red}} \text{ and } y \in V'_{\text{blue}} \text{ such that } xy \in \partial \mathcal{R} \cap \partial \mathcal{B}. \] (1)

Indeed, any such edge \(xy\) constitutes an active pair (by Lemma 2.3), and as \(|V_{\text{red}}| > \delta t + 2\), there must be a vertex \(z \in V'_{\text{red}}\) such that \(xyz\). This yields a contradiction with the maximality of the matching \(\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}\).

We show that if \(|V'_{\text{red}}|\) and \(|V'_{\text{blue}}|\) are both greater than \(2\delta t\), then we can find a pair \(xy\) contradicting (1). So we can assume, by symmetry of the argument, that \(V'_{\text{blue}}\) is negligible in \(\mathcal{K}\).

Next, because of the maximality of \(\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}\), each edge having all its vertices in \(V'_{\text{red}}\) is blue. By Lemma 2.3, \(V'_{\text{red}}\) is negligible in \(\mathcal{K}\) (in which case we are done), or \(V'_{\text{red}}\) is sufficiently large to induce a dense monochromatic blue component \(\mathcal{B}'\) such that \(V'_{\text{red}} \setminus V(\mathcal{B}')\) is negligible in \(\mathcal{K}\) and satisfying the hypothesis of Theorem 2.2. Therefore, the blue component \(\mathcal{B}'\) contains a perfect matching.

At this point, we have three disjoint monochromatic connected matchings, one in red (\(\mathcal{M}_{\text{red}} \subseteq \mathcal{R}\)) and two in blue (\(\mathcal{M}_{\text{blue}} \subseteq \mathcal{B} \text{ and } \mathcal{M}'_{\text{blue}} \subseteq \mathcal{B}'\)). Together, these matchings cover all but a negligible set of vertices in \(\mathcal{K}\). Notice that \(\mathcal{B}\) and \(\mathcal{B}'\) can not be the same component because of the maximality of \(\mathcal{M}_{\text{red}} \cup \mathcal{M}_{\text{blue}}\).

Our aim now is to dissolve the blue matching \(\mathcal{M}_{\text{blue}}\) and cover all but a negligible set (in \(\mathcal{K}\)) of \(V(\mathcal{M}_{\text{blue}})\) with edges in \(\mathcal{R}\). To this end, we show that \(V(\mathcal{M}_{\text{blue}})\) is negligible in \(\mathcal{K}\) (in which case we are done) or, as a consequence of Lemma 2.3, \(V(\mathcal{M}_{\text{blue}})\) is contained in \(V_{\text{red}}\). Finally, by using the defect form of Hall’s theorem, we cover the vertices of \(\mathcal{M}_{\text{blue}}\) with a matching \(\mathcal{M}'_{\text{red}}\) in \(\partial \mathcal{R}\). In other words, \(\mathcal{M}_{\text{red}} \cup \mathcal{M}'_{\text{red}}\) and \(\mathcal{M}'_{\text{blue}}\) are the two monochromatic connected matchings we had to find.

References


